SORTABLE ELEMENTS AND TORSION PAIRS FOR QUIVERS

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Abstract. We explain a close relationship between c-sortable elements and torsion pairs. In particular, we give an explicit description of the cofinite torsion classes in the context of the Coxeter group. This result allows a proof of some conjectures proposed by [9].

1. Introduction

Throughout the paper, let K be an algebraically closed field, Q a finite acyclic quiver and W the Coxeter group of Q. Path algebra KQ is one of the most fundamental classes of finite dimensional algebras. Recently, Oppermann-Reiten-Thomas gave the following strong connection between the representation theory of KQ and combinatorics of W as follows.

Theorem 1. [9] *Let* $\mathrm{cof.quot}KQ$ *be the set of cofinite (i.e.,there are only finitely many indecomposable modules which are not in the category) quotient closed subcategories of* $\mathrm{mod}KQ$*. Then there is a bijection*

$$W \longrightarrow \mathrm{cof.quot}KQ.$$

We denote by $\mathcal{C}_w$ the corresponding category for $w \in W$. Our investigation has one of its primary origins in the following natural questions and the related conjectures posed in [9, Conjecture 11.1,11.2].

Question.

(a) When is $\mathcal{C}_w$ a torsion class of $\mathrm{mod}KQ$ for $w \in W$?

(b) When $\mathcal{C}_w$ is a torsion class, how can we relate w to a c-sortable element x which provides the corresponding finite torsion free class ? (see also Corollary 17)

To give answers, we use the theory of preprojective algebras. Preprojective algebras are a fundamental class of algebras. Recently, close links have been discovered between preprojective algebras and Coxeter groups [6, 4, 7, 2, 8], and the representation theory of preprojective algebras can be viewed as providing a categorification of the structure of the corresponding Coxeter group (see Theorem 8).

2. Preliminary

Notation. For a K-algebra Λ, we denote by $\mathrm{mod}\Lambda$ the category of finite dimensional right Λ-modules. Let X be a Λ-module. We denote by $\mathsf{add}X$ (respectively, $\mathsf{Sub}X$, $\mathsf{Fac}X$) the full subcategory whose objects are direct summands (respectively, submodules, factor modules) of finite direct sums of copies of X.

The detailed version of this paper will be submitted for publication elsewhere.

In this section, we recall some definitions and important properties. Throughout this note, let Q be a finite connected acyclic quiver with vertices $Q_0 = \{1, \dots, n\}$. We always assume for simplicity that Q_0 are admissibly numbered, that is, we have an arrow $j \to i$, then $j < i$.

2.1. **Coxeter groups.**

Definition 2. The *Coxeter group* W associated to Q is defined by the generators $S := \{s_1, \dots, s_n\}$ and relations

- $s_i^2 = 1$,
- $s_i s_j = s_j s_i$ if there is no arrow between i and j in Q,
- $s_i s_j s_i = s_j s_i s_j$ if there is precisely one arrow between i and j in Q.

We denote by $\mathbf{w}$ a word, that is, an expression in the free monoid generated by s_i for $i \in Q_0$ and w its equivalence class in the Coxeter group W. We denote by $\leq$ the (*right*) *weak order*. An element $c = s_{u_1} \dots s_{u_l}$ is called a *Coxeter element* if $l = n$ and $\{u_1, \dots, u_l\} = \{1, \dots, n\}$. A Coxeter element $c = s_1 \dots s_n$ is called *admissible* with respect to the orientation of Q. In this note, by a Coxeter element, we mean an admissible one.

Definition 3. Let c be a Coxeter element. Fix a reduced expression of c and regard c as a reduced word. For $w \in W$, we denote the support of W by *supp*(w), that is, the set of generators occurring in a reduced expression of w.

We call an element $w \in W$ *c-sortable* if there exists a reduced expression of w of the form $\mathbf{w} = c^{(0)} c^{(1)} \dots c^{(m)}$, where all $c^{(t)}$ are subwords of c whose supports satisfy

$$\mathit{supp}(c^{(m)}) \subseteq \mathit{supp}(c^{(m-1)}) \subseteq \dots \subseteq \mathit{supp}(c^{(1)}) \subseteq \mathit{supp}(c^{(0)}) \subseteq Q_0.$$

For the generators $S = \{s_1, \dots, s_n\}$, we let $\langle s \rangle := S \setminus \{s\}$ and denote $W_{\langle s \rangle}$ by the subgroup of W generated by $\langle s \rangle$. For any $w \in W$, there is a unique factorization $w = w_{\langle s \rangle} \cdot w^{\langle s \rangle}$ maximizing $\ell(w_{\langle s \rangle})$ for $w_{\langle s \rangle} \in W_{\langle s \rangle}$ and $\ell(w_{\langle s \rangle}) + \ell(w^{\langle s \rangle}) = \ell(w)$.

Then we give the following map introduced by Reading [10].

Definition 4. Let c be a Coxeter element and let s be initial in c. Then, define $\pi^c(id) = id$ and, for each $w \in W$, we define

$$\pi^c(w) := \begin{cases} s\pi^{scs}(sw) & \text{if } \ell(sw) < \ell(w) \\ \pi^{sc}(w_{\langle s \rangle}) & \text{if } \ell(sw) > \ell(w). \end{cases}$$

Then this has gives the following property.

Theorem 5. [11, Proposition 3.2][13, Corollary 6.2] *For any $w \in W$, $\pi^c(w)$ is the unique maximal c-sortable element below w in the weak order.*

Example 6. Let Q be the following quiver

$$1 \to 2 \to 3.$$

Then $c = s_1 s_2 s_3$. For example $s_1 s_2 s_3 s_2$ is a c-sortable element, and $s_1 s_2 s_3 s_2 s_1$ is not.

Let $w = s_1 s_2 s_3 s_2 s_1$. Then one can check that $\pi^c(w) = s_1 s_2 s_3 s_2$ and it is a unique maximal c-sortable element below w.

2.2. **Preprojective algebras.** Next we discuss a relationship between preprojective algebras and the Coxeter groups.

Definition 7. We denote by Q_1 the set of arrows of a quiver Q. The preprojective algebra associated to Q is the algebra

$$\Lambda = K\overline{Q}/\langle \sum_{a \in Q_1} (aa^* - a^*a) \rangle$$

where $\overline{Q}$ is the double quiver of Q, which is obtained from Q by adding for each arrow $a : i \to j$ in Q_1 an arrow $a^* : i \leftarrow j$ pointing in the opposite direction.

Let Λ the preprojective algebra of Q. We denote by I_i the two-sided ideal of Λ generated by $1 - e_i$, where e_i is a primitive idempotent of Λ for $i \in Q_0$. We denote by $\langle I_1, \ldots, I_n \rangle$ the set of ideals of Λ which can be written as $I_{u_l} \cdots I_{u_1}$ for some $l \geq 0$ and $u_1, \ldots, u_l \in Q_0$. Then we have the following result (see also [7, Theorem 2.14] in the case of Dynkin).

Theorem 8. [4, Theorem III.1.9] *There exists a bijection $W \to \langle I_1, \ldots, I_n \rangle$. It is given by $w \mapsto I_w = I_{u_l} \cdots I_{u_1}$ for any reduced expression $w = s_{u_1} \cdots s_{u_l}$.*

Note that the product of ideals is taken in the opposite order to the product of expression of w. This is just because we follow the convention of [9, 3].

Next we briefly recall main results of [9], which give a deep connection between path algebras, preprojective algebras and the Coxeter groups.

Definition 9. Let Λ be the preprojective algebra of Q. For a Λ-module X, we denote by X_{KQ} the KQ-module by the restriction, that is, we forget the action of the arrows $a^* \in \overline{Q}$. Moreover we associate the subcategory

$$\mathsf{res}X = \mathsf{add}X_{KQ} \bigcap \mathsf{mod}KQ.$$

In the case of non-Dynkin, we denote by $\overline{\mathsf{res}X}$ the additive category generated by $\mathsf{res}X$ together with all non-preprojective indecomposable KQ-modules.

Then we can give a more precise formulation of Theorem 1 as follows.

Theorem 10. [9] *The map $w \mapsto \overline{\mathsf{res}I_w}$ gives a bijection between the elements of W and the cofinite (additive) quotient closed subcategories of* $\mathrm{mod}KQ$*.*

Example 11. Let $Q = (1 \to 2 \to 3)$. Then the AR quiver of $\mathrm{mod}KQ$ is given by

$$\begin{tikzcd}[column sep=small, row sep=small] & & {\substack{3\\2\\1}} \arrow[dr] & & \\ & {\substack{2\\1}} \arrow[ur] \arrow[dr] & & {\substack{3\\2}} \arrow[dr] & \\ 1 \arrow[ur] & & 2 \arrow[ur] & & 3 \end{tikzcd}.$$

For example, take $w = s_1 s_3$. Then, one can check that $\mathsf{res}I_w = \mathsf{add}\{2, 3, {\substack{2\\1}}, {\substack{3\\2}}\}$.

Particularly important class of quotient subcategories is a torsion class. Therefore, it is quite important to ask when $\overline{\mathsf{res}I_w}$ is a torsion class. Our aim is to give an answer to this question.

2.3. **Sortable elements and finite torsion-free classes.** Next we recall the notion of support tilting modules.

Definition 12. [5] For a KQ-module X, we call X *support tilting* if there exists an idempotent e of KQ such that X is a tilting $(KQ/\langle e\rangle)$-module.

Then we have the following result (see also [1, Theorem 2.7]).

Theorem 13. [5, Theorem 2.11] *Let Q be an acyclic quiver. There is a bijection between the set* $\mathrm{stilt}KQ$ *of isomorphism classes of basic support tilting KQ-modules and the set* $\mathrm{fin.tors}KQ$ *of functorially finite torsion classes of* $\mathsf{mod}KQ$. *It is given by* $\mathrm{stilt}KQ \ni T \mapsto \mathsf{Fac}T \in \mathrm{fin.tors}KQ$ *and* $\mathrm{fin.tors}KQ \ni \mathcal{T} \mapsto P(\mathcal{T}) \in \mathrm{stilt}KQ$, *where $P(\mathcal{T})$ denote the direct sum of one copy of each of the indecomposable Ext-projective of $\mathcal{T}$ up to isomorphism.*

We also recall layers following [3]. For any reduced word $\mathbf{w} = s_{u_1} \dots s_{u_l}$, we have the chain of ideals

$$\Lambda \supset I_{u_1} \supset I_{u_2}I_{u_1} \supset \dots \supset I_{u_l} \dots I_{u_2}I_{u_1} = I_{\mathbf{w}}.$$

For $j = 1, \dots, l$, we define the *layer*

$$L_{\mathbf{w}}^j = e_{u_j} L_{\mathbf{w}}^j := \frac{I_{u_{j-1}} \dots I_{u_1}}{I_{u_j} \dots I_{u_1}}.$$

Note that any layer $L_{\mathbf{w}}^j$ is an indecomposable Λ-module for any $j = 1, \dots, l$ [3].

Then, for a c-sortable word, we can give a support tilting KQ-module and the associated torsion-free class, which can be explicitly described by layers, as follows.

Theorem 14. [3, Theorem 3.3, 3.11 and Corollary 3.10] *Let c be a Coxeter element of Q and $\mathbf{w} = c^{(0)}c^{(1)} \dots c^{(m)} = s_{u_1} \dots s_{u_l}$ a c-sortable word.*

(a) *$L_{\mathbf{w}}^j$ is a non-zero indecomposable KQ-module for all $j = 1, \dots, l$.*

Moreover, we denote by $Q^{(0)}$ the quiver Q restricted to the support of $c^{(0)}$. For $i \in Q_0^{(0)}$, we denote by $t_{\mathbf{w}}(i)$ the maximal integer such that $u_{t_{\mathbf{w}}(i)} = i$ and let

$$T_{\mathbf{w}} := \bigoplus_{i \in Q_0^{(0)}} L_{\mathbf{w}}^{t_{\mathbf{w}}(i)}.$$

(b) *$T_{\mathbf{w}}$ is a tilting $KQ^{(0)}$-module, that is, support tilting KQ-module.*

(c) *We have* $\mathsf{Sub}T_{\mathbf{w}} = \mathsf{add}\{L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^l\} = \mathsf{res}(\Lambda/I_w)$.

Example 15. Let Q be the following quiver

$$\begin{tikzcd}[row sep=small, column sep=small] & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3. \end{tikzcd}$$

Then $s_1s_2s_3$ is a Coxeter element of Q. Let $\mathbf{w} = s_1s_2s_3s_1s_2s_1$. Then we have

$$L_{\mathbf{w}}^1 = {\scriptstyle 1}, \ L_{\mathbf{w}}^2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \ L_{\mathbf{w}}^3 = \begin{smallmatrix} & 3 & & \\ 1 & & 2 & \\ & & & 1 \end{smallmatrix}, \ L_{\mathbf{w}}^4 = \begin{smallmatrix} 2 & & 3 & & \\ & 1 & & 2 & \\ & & & & 1 \end{smallmatrix}, \ L_{\mathbf{w}}^5 = \begin{smallmatrix} & 3 & & & & \\ 1 & & 2 & & 3 & \\ & & & 1 & & 2 & \\ & & & & & & 1 \end{smallmatrix}, \ L_{\mathbf{w}}^6 = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}.$$

Hence we have $T_{\mathbf{w}} = \begin{smallmatrix} & 3 & & \\ 1 & & 2 & \\ & & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & & & & \\ 1 & & 2 & & 3 & \\ & & & 1 & & 2 & \\ & & & & & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$ and

$$\mathsf{Sub}T_{\mathbf{w}} = \mathsf{add}\{L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^l\} = \mathsf{res}(\Lambda/I_w).$$

We call a torsion-free class *finite* if it has finitely many indecomposable modules. Theorem 14 implies that a c-sortable element gives a support tilting module and the finite torsion-free class associated with it. Conversely, any finite torsion-free classes of $\mathrm{mod}\, KQ$ is given by a support tilting module induced by a c-sortable element as follows.

Theorem 16. [3, Theorem 3.16] *Let $\mathcal{F}$ be a finite torsion-free class. Then there exists a unique c-sortable word $\mathbf{w}$ such that $T_{\mathbf{w}}$ is a support tilting KQ-module and $\mathcal{F} = \mathsf{Sub}T_{\mathbf{w}}$.*

Then, combining with Theorems 14 and 16, we provide the following correspondence.

Corollary 17. [3, Corollary 3.18] *The map $w \mapsto \mathsf{res}(\Lambda/I_w)$ gives a bijection*

$$\{c\text{-sortable elements}\} \longleftrightarrow \{\text{finite torsion-free classes of } \mathrm{mod}KQ\}.$$

3. Our results

Using the above results, we discuss a connection between torsion pairs of $\mathrm{mod}KQ$ and W. Let Q be a finite acyclic quiver, Λ the preprojective algebra of Q, W the Coxeter group of Q and c the Coxeter element.

First, we introduce the following definition.

Definition 18. A c-sortable element x is called *bounded* if there exists a positive integer N such that $x \leq c^N$. In the Dynkin case, we regard any c-sortable element as bounded. We denote by wc-sort$W$ the set of bounded c-sortable elements.

Example 19. (a) Let Q be the following quiver

$$1 \to 2 \rightrightarrows 3.$$

Because

$$\begin{aligned} c^3 &= s_1s_2s_3s_1s_2s_3s_1s_2s_3 \\ &= s_1s_2s_3s_1s_2s_1s_3s_2s_3 \\ &= s_1s_2s_3s_2s_1s_2s_3s_2s_3, \end{aligned}$$

we have $s_1s_2s_3s_2 \leq c^3$ and hence $s_1s_2s_3s_2$ is bounded c-sortable.

(b) Let Q be the following quiver

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3. \end{tikzcd}$$

Then one can check that $s_1s_2s_3s_2$ is not bounded c-sortable.

Then we give the following lemma.

Lemma 20. *Let x be a c-sortable element. Then the following are equivalent.*

(a) *x is bounded c-sortable.*
(b) *Any module of $\mathsf{res}(\Lambda/I_x)$ is a preprojective module.*
(c) *The corresponding torsion class ${}^{\perp}(\mathsf{res}(\Lambda/I_x))$ is cofinite.*

Proof. We only consider the non-Dynkin case.

Then we have $\mathsf{res}(\Lambda/I_{c^N}) = \mathsf{add}\{KQ, \tau^-(KQ), \ldots, \tau^{-N}(KQ)\}$. On the other hand, we have $x \leq c^N$ if and only if $\mathsf{res}(\Lambda/I_x) \subset \mathsf{res}(\Lambda/I_{c^N})$. Thus it implies the the equivalence of (a) and (b). The equivalence of (b) and (c) is straightforward from the structure of the AR quiver. $\square$

Thus, bounded c-sortable elements are essential objects from the viewpoint of the question. To give answers to our question, we also introduce the following terminology.

Definition 21. Let x be a c-sortable element. If there exists a maximum element amongst $w \in W$ satisfying $\pi^c(w) = x$, then we denote it by $\widehat{x}^c = \widehat{x}$ and call it *c-antisortable*, following the definition from [12]. We denote by c-comp$W$ the set of c-antisortable elements of W.

Example 22. (a) Let Q be the following quiver

$$1 \to 2 \rightrightarrows 3.$$

Take a c-sortable element $x = s_1s_2s_3s_2$. Then one can check that $\widehat{x} = s_1s_2s_3s_2s_1$.

(b) Let Q be the following quiver

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3. \end{tikzcd}$$

Take a c-sortable element $x = s_1s_2s_3s_2$. Consider the following infinite word

$$s_1s_2s_3s_2s_1s_3s_2s_1s_3s_2s_1s_3s_2\cdots.$$

Then from the word, we can take an arbitrary longer element w such that $\pi^c(w) = x$. Thus, $\widehat{x}$ does not exist.

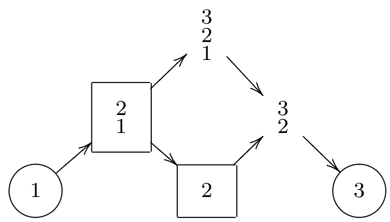
Then our main result is following theorem.

Theorem 23. *We have the following bijections:*

$$\begin{tikzcd}[column sep=small] & \{\text{cofinite torsion pairs of } \operatorname{mod} KQ\} \arrow[dl, "\overline{\mathsf{res} I_{(-)}}"'] \arrow[dr, "\mathsf{res}(\Lambda/I_{(-)})"] & \\ c\text{-comp}W \arrow[ur, "{(\overline{\mathsf{res} I_{(-)}}, \mathsf{res}(\Lambda/I_{(-)}))}"] \arrow[rr, shift left, "\pi^c(-)"] & & \mathrm{w}c\text{-sort}W \arrow[ul] \arrow[ll, shift left, "\widehat{(-)}"] \end{tikzcd}$$

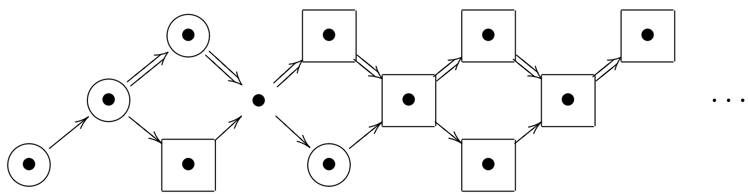
Here we call a torsion pair *cofinite* if the torsion class is cofinite. Thus we can give a answer to the previous question, confirming a conjecture of [9, Conjecture 11.1]: the cofinite quotient-closed category $\overline{\mathsf{res} I_w}$ is a torsion class precisely if w is c-antisortable. Moreover, in this case the corresponding torsion free class is the one associated to $\pi^c(w)$, again confirming a conjecture of [9, Conjecture 11.2]. Thus the theorem implies that these torsion pairs can be completely controlled by bounded c-sortable elements and c-antisortable elements.

Example 24. (a) Let $Q = (1 \to 2 \to 3)$. Then the AR quiver of $\mathrm{mod} KQ$ is given by



For example, we take a c-sortable element $x = s_1 s_3$. Then we have the torsion-free class $\mathsf{res}(\Lambda/I_x) = \mathsf{add}\{\,1\,,\,3\,\}$ whose modules are circled above. Then one can check that the corresponding torsion class is $\mathsf{add}\{\begin{smallmatrix}2\\1\end{smallmatrix}, 2\}$ whose modules are squared above. By Theorem 10, it is given as $\mathsf{res} I_w$ for $w = s_1 s_3 s_2 s_1$. Then the theorem implies that this element gives $\widehat{x}$.

(b) Let $Q = (1 \to 2 \Rightarrow 3)$. Then the preprojective component of the AR quiver of mod KQ is given as the translation quiver. Thus it is given as the form



For example, we take a c-sortable element $x = s_1 s_2 s_3 s_2$, which is bounded c-sortable. Then $\mathsf{res}(\Lambda/I_x)$ consists of the modules which are circled above. The corresponding torsion class consists of the modules which are squared above and all the rest. It is given as $\overline{\mathsf{res} I_w}$ for $w = s_1 s_2 s_3 s_2 s_1$. Therefore our theorem implies that we have $\widehat{x} = s_1 s_2 s_3 s_2 s_1$.

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